

# A FAMILY OF DIGIT FUNCTIONS WITH LARGE PERIODS

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**ABSTRACT.** For odd  $n \geq 3$ , we consider a general hypothetical identity for the differences  $S_{n,0}(x)$  of multiples of  $n$  with even and odd digit sums in the base  $n-1$  in interval  $[0, x)$ , which we prove in the cases  $n = 3$  and  $n = 5$  and empirically confirm for some other  $n$ . We give a verification algorithm for this identity for any odd  $n$ . The hypothetical identity allows to give a general recursion for  $S_{n,0}(x)$  for every integer  $x$  depending on the residue of  $x$  modulo  $p(n) = 2n(n-1)^{n-1}$ , such that  $p(3) = 24$ ,  $p(5) = 2560$ ,  $p(7) = 653184$ , etc.

## 1. INTRODUCTION

For  $x \in \mathbb{N}$  and  $n \geq 3$ , denote by  $S_n(x)$  the sum

$$(1) \quad S_{n,j}(x) = \sum_{0 \leq r < x: r \equiv j \pmod{n}} (-1)^{s_{n-1}(r)},$$

where  $s_{n-1}(r)$  is the digit sum of  $r$  in base  $n-1$ .

Note that, in particular,  $S_{3,0}(x)$  equals the difference between the numbers of multiples of 3 with even and odd binary digit sums (or multiples of 3 from sequences A001969 and A000069 in [7]) in interval  $[0, x)$ .

Leo Moser (cf. [3], Introduction) conjectured that always

$$(2) \quad S_{3,0}(x) > 0.$$

Newman [3] proved this conjecture. Moreover, he obtained the inequalities

$$(3) \quad \frac{1}{20} < S_{3,0}(x)x^{-\lambda} < 5,$$

where

$$(4) \quad \lambda = \frac{\ln 3}{\ln 4} = 0.792481\dots$$

In connection with this, the qualitative result (2) we call a weak Newman phenomenon (or Moser-Newman phenomenon), while an estimating result of the form (3) we call a strong Newman phenomenon.

In 1983, Coquet [1] studied a very complicated continuous and nowhere differentiable fractal function  $F(x)$  with period 1 for which

$$(5) \quad S_{3,0}(3x) = x^\lambda F\left(\frac{\ln x}{\ln 4}\right) + \frac{\eta(x)}{3},$$

where

$$(6) \quad \eta(x) = \begin{cases} 0, & \text{if } x \text{ is even,} \\ (-1)^{s_2(3x-1)}, & \text{if } x \text{ is odd.} \end{cases}$$

He obtained that

$$(7) \quad \limsup_{x \rightarrow \infty, x \in \mathbb{N}} S_{3,0}(3x)x^{-\lambda} = \frac{55}{3} \left(\frac{3}{65}\right)^\lambda = 1.601958421 \dots,$$

$$(8) \quad \liminf_{x \rightarrow \infty, x \in \mathbb{N}} S_{3,0}(3x)x^{-\lambda} = \frac{2\sqrt{3}}{3} = 1.154700538 \dots$$

In 2007, Shevelev [4] gave an elementary proof of Coquet's formulas (7)-(8) and his sharp estimates in the form

$$(9) \quad \frac{2\sqrt{3}}{3}x^\lambda \leq S_{3,0}(3x, 0) \leq \frac{55}{3} \left(\frac{3}{65}\right)^\lambda x^\lambda, \quad x \in \mathbb{N}.$$

In [4] it was found the following simple identity

$$(10) \quad S_{3,0}(4x) = 3S_{3,0}(x), \text{ where } x \text{ is even.}$$

Since in the left hand side of (10) the argument  $4x \equiv 0 \pmod{8}$  then (10) is not a recursion for evaluation of  $S_{3,0}(x)$ . However, in the same work Shevelev found the following recursion for fast calculation of  $S_{3,0}(x)$  :

$$(11) \quad S_{3,0}(x) = 3S_{3,0}\left(\left\lfloor \frac{x}{4} \right\rfloor\right) + \nu(x),$$

where

$$(12) \quad \nu(x) = \begin{cases} 0, & \text{if } x \equiv 0, 7, 8, 9, 16, 17, 18, 22, 23 \pmod{24}; \\ (-1)^{s_2(x)}, & \text{if } x \equiv 3, 4, 10, 12, 20 \pmod{24}; \\ (-1)^{s_2(x)+1}, & \text{if } x \equiv 1, 2, 5, 6, 11, 19, 21 \pmod{24}; \\ 2(-1)^{s_2(x)}, & \text{if } x \equiv 15 \pmod{24}; \\ 2(-1)^{s_2(x)+1}, & \text{if } x \equiv 13, 14 \pmod{24}. \end{cases}$$

In 2008, Drmota and Stoll [2] proved a generalized weak Newman phenomenon, showing that (2) is valid for  $S_{n,0}(x)$  for every  $n \geq 3$ , at least beginning with  $x \geq x_0(n)$ . A year before, Shevelev [5] proved a strong form

of this generalization, but yet only in "full" intervals of the form  $[0, (n-1)^{2p})$ . Recently Shevelev and Moses [6] in the case of odd  $n \geq 3$  and  $p \geq \frac{n-1}{2}$  found the relation

$$(13) \quad \sum_{k=0}^{\frac{n-1}{2}} (-1)^k \binom{n}{2k} S_{n,0}((n-1)^{2p-2k}) = \begin{cases} 0, & \text{if } p \geq \frac{n+1}{2}, \\ (-1)^n, & \text{if } p = \frac{n-1}{2}. \end{cases}$$

In the case of  $p = \frac{n-1}{2}$ , (13) could be rewrite in the form

$$(14) \quad \sum_{j=0}^{\frac{n-1}{2}} (-1)^j \binom{n}{2j+1} S_{n,0}((n-1)^{2j}) = 1.$$

Numerous experiments show that, most likely, the following more general relation takes place:

$$(15) \quad \sum_{j=0}^{\frac{n-1}{2}} (-1)^j \binom{n}{2j+1} S_{n,0}((n-1)^{2j}x) = \sum_{j=0}^{n-1} S_{n,j}(x), \quad x \geq 1, \quad n \equiv 1 \pmod{2}.$$

In particular, we verified (15) for  $n = 3, 5, 7, \dots, 35$  and  $1 \leq x \leq 1000$ . It is clear that (14) is a special case of (15) for  $x = 1$ , since

$$(16) \quad S_{n,j}(1) = \begin{cases} 1, & \text{if } j = 0, \\ 0, & \text{if } 1 \leq j \leq n-1. \end{cases}$$

Below we show that (15) allows with the uniform positions to find a recursion for  $S_{n,0}(x)$  for every odd  $n \geq 3$ . In the two first sections we prove identity (15) in cases  $n = 3$  and  $n = 5$ . In Section 4 we give a general verification algorithm for the identity (15) which allows to prove the identity (15) for  $n = 7, 9, \dots, \text{etc.}$  In Section 5 we give a simplification of the conjectural equality (15). In Section 6 we prove the recursion in case  $n = 3$  and in Section 7 we give the recursion in case  $n = 5$ . After these sections, in supposition that (15) is true, it will be clear how to find the further recursions for odd  $n \geq 7$ .

## 2. THE IDENTITY IN CASE $n = 3$

Note that, by (1),

$$S_{3,j}(x) = \sum_{0 \leq r < x: \quad r \equiv j \pmod{3}} (-1)^{s_2(r)}$$

which yields that

$$(17) \quad \sum_{0 \leq r < 2x: \quad r \equiv 2j \pmod{6}} (-1)^{s_2(r)}, \quad j = 0, 1, 2.$$

On the other hand,

$$(18) \quad S_{3,j}(2x) = \sum_{0 \leq r < 2x: \substack{r \equiv j \\ (\text{mod } 6)}} (-1)^{s_2(r)} + \sum_{0 \leq r < 2x: \substack{r \equiv j+3 \\ (\text{mod } 6)}} (-1)^{s_2(r)}, \quad j = 0, 1, 2.$$

Using (18), for  $j = 0, 1, 2$ , we consecutively find

$$(19) \quad S_{3,0}(2x) = \sum_{0 \leq r < 2x: \substack{r \equiv 0 \\ (\text{mod } 6)}} (-1)^{s_2(r)} - \sum_{0 \leq r < 2x: \substack{r \equiv 2 \\ (\text{mod } 6)}} (-1)^{s_2(r)},$$

$$(20) \quad S_{3,1}(2x) = - \sum_{0 \leq r < 2x: \substack{r \equiv 0 \\ (\text{mod } 6)}} (-1)^{s_2(r)} + \sum_{0 \leq r < 2x: \substack{r \equiv 4 \\ (\text{mod } 6)}} (-1)^{s_2(r)},$$

$$(21) \quad S_{3,2}(2x) = \sum_{0 \leq r < 2x: \substack{r \equiv 2 \\ (\text{mod } 6)}} (-1)^{s_2(r)} - \sum_{0 \leq r < 2x: \substack{r \equiv 4 \\ (\text{mod } 6)}} (-1)^{s_2(r)}.$$

Now the application of (17) to (19)-(21) yields the relations

$$(22) \quad S_{3,0}(2x) = S_{3,0}(x) - S_{3,1}(x),$$

$$(23) \quad S_{3,1}(2x) = -S_{3,0}(x) + S_{3,2}(x),$$

$$(24) \quad S_{3,2}(2x) = S_{3,1}(x) - S_{3,2}(x).$$

For  $n = 3$ , the left hand side of (15) is  $3S_{3,0}(x) - S_{3,0}(4x)$  and, using (22)-(24), we have

$$\begin{aligned} 3S_{3,0}(x) - S_{3,0}(4x) &= 3S_{3,0}(x) - S_{3,0}(2x) + S_{3,1}(2x) = \\ &= 3S_{3,0}(x) - S_{3,0}(x) + S_{3,1}(x) - S_{3,0}(x) + S_{3,2}(x) = \\ &= S_{3,0}(x) + S_{3,1}(x) + S_{3,2}(x) \end{aligned}$$

which proves (15) in the case  $n = 3$ .

3. THE IDENTITY IN CASE  $n = 5$ 

In the same way, instead of (22)-(24), we find the following relations

$$(25) \quad S_{5,0}(4x) = S_{5,0}(x) - S_{5,1}(x) + S_{5,2}(x) - S_{5,3}(x),$$

$$(26) \quad S_{5,1}(4x) = -S_{5,0}(x) + S_{5,1}(x) - S_{5,2}(x) + S_{5,4}(x),$$

$$(27) \quad S_{5,2}(4x) = S_{5,0}(x) - S_{5,1}(x) + S_{5,3}(x) - S_{5,4}(x),$$

$$(28) \quad S_{5,3}(4x) = -S_{5,0}(x) + S_{5,2}(x) - S_{5,3}(x) + S_{5,4}(x),$$

$$(29) \quad S_{5,4}(4x) = S_{5,1}(x) - S_{5,2}(x) + S_{5,3}(x) - S_{5,4}(x).$$

For  $n = 5$ , the left hand side of (15) is

$$(30) \quad 5S_{5,0}(x) - 10S_{5,0}(16x) + S_{5,0}(256x).$$

Using (25)-(29), we easily find

$$(31) \quad S_{5,0}(16x) = 4S_{5,0}(x) - 3S_{5,1}(x) + S_{5,2}(x) + S_{5,3}(x) - 3S_{5,4}(x),$$

$$(32) \quad S_{5,1}(16x) = -3S_{5,0}(x) + 4S_{5,1}(x) - 3S_{5,2}(x) + S_{5,3}(x) + S_{5,4}(x),$$

$$(33) \quad S_{5,2}(16x) = S_{5,0}(x) - 3S_{5,1}(x) + 4S_{5,2}(x) - 3S_{5,3}(x) + S_{5,4}(x),$$

$$(34) \quad S_{5,3}(16x) = S_{5,0}(x) + S_{5,1}(x) - 3S_{5,2}(x) + 4S_{5,3}(x) - 3S_{5,4}(x),$$

$$(35) \quad S_{5,4}(16x) = -3S_{5,0}(x) + S_{5,1}(x) + S_{5,2}(x) - 3S_{5,3}(x) + 4S_{5,4}(x).$$

Now using (31)-(35), we find

$$(36) \quad \begin{aligned} S_{5,0}(256x) &= 36S_{5,0}(x) - 29S_{5,1}(x) + \\ &11S_{5,2}(x) + 11S_{5,3}(x) - 29S_{5,4}(x). \end{aligned}$$

Finally, for the expression (30), using (31) and (36), we have

$$(37) \quad \begin{aligned} 5S_{5,0}(x) - 10S_{5,0}(16x) + S_{5,0}(256x) &= \\ S_{5,0}(x) + S_{5,1}(x) + S_{5,2}(x) + S_{5,3}(x) + S_{5,4}(x). \end{aligned}$$

It is the identity (15) in the case  $n = 5$ .

In particular, for  $x = 1$ , we again have (14). Note that (41) means that its left hand side taken with sign  $(-1)^{s_{n-1}(x-1)}$  is periodic with period 2:

$$\begin{aligned}
 & (-1)^{s_{n-1}(x-1)} \sum_{j=0}^{\frac{n-1}{2}} (-1)^j \binom{n}{2j+1} S_{n,0}((n-1)^{2j}x) = \\
 (42) \quad & \begin{cases} 0, & \text{if } x \text{ is even,} \\ 1, & \text{if } x \text{ is odd.} \end{cases}
 \end{aligned}$$

## 6. RECURSION FOR $S_{3,0}(x)$

Here we prove (11)-(12). Let us write (42) for  $n = 3$  and  $x := \lfloor \frac{x}{4} \rfloor$ . We have

$$\begin{aligned}
 & (-1)^{s_2(\lfloor \frac{x}{4} \rfloor - 1)} (3S_{3,0}(\lfloor \frac{x}{4} \rfloor) - S_{3,0}(4\lfloor \frac{x}{4} \rfloor)) = \\
 (43) \quad & \begin{cases} 0, & \text{if } \lfloor \frac{x}{4} \rfloor \text{ is even,} \\ 1, & \text{if } \lfloor \frac{x}{4} \rfloor \text{ is odd.} \end{cases}
 \end{aligned}$$

Note that  $\lfloor \frac{x}{4} \rfloor$  is even, if  $x = 0, 1, 2, 3, 8, 9, 10, 11, \dots$  and odd for other integers. Thus we obtain

**Lemma 1.** *The sequence  $\{A_3(x)\}$ , where*

$$(44) \quad A_3(x) = (-1)^{s_2(\lfloor \frac{x}{4} \rfloor - 1)} (3S_{3,0}(\lfloor \frac{x}{4} \rfloor) - S_{3,0}(4\lfloor \frac{x}{4} \rfloor)),$$

*is periodic with the period 8, such that*

$$(45) \quad A_3(x) = \begin{cases} 0, & \text{if } x \equiv 0, 1, 2, 3, \pmod{8}, \\ 1, & \text{if } x \equiv 4, 5, 6, 7 \pmod{8}. \end{cases}$$

Consider the difference

$$(46) \quad \Delta_3(x) = S_{3,0}(x) - S_{3,0}(4\lfloor \frac{x}{4} \rfloor).$$

**Lemma 2.** *We have*

$$(47) \quad \Delta_3(x) = \begin{cases} (-1)^{s_2(x-1)}, & \text{if } x \equiv 1, 7 \text{ or } 10 \pmod{12} \\ (-1)^{s_2(x-2)}, & \text{if } x \equiv 2 \text{ or } 11 \pmod{12} \\ (-1)^{s_2(x-3)}, & \text{if } x \equiv 3 \pmod{12} \\ 0, & \text{otherwise.} \end{cases}$$

*Proof.* Let  $x = 12t + j$ ,  $j = 0, 1, \dots, 11$ . Consider 3 cases.

a)  $j = 0, 1, 2$  or  $3$ .

Then

$$\begin{aligned}
 \Delta_3(x) &= S_{3,0}(12t + j) - S_{3,0}(12t) = \\
 & \begin{cases} 0, & \text{if } j = 0, \\ (-1)^{s_2(x-j)}, & \text{if } j = 1, 2, 3. \end{cases}
 \end{aligned}$$

b)  $j = 4, 5, 6$  or  $7$ .

Then

$$\Delta_3(x) = S_{3,0}(12t + j) - S_{3,0}(12t + 4) =$$

$$\begin{cases} 0, & \text{if } j = 4, 5, 6, \\ (-1)^{s_2(x-1)}, & \text{if } j = 7. \end{cases}$$

c)  $j = 8, 9, 10$  or  $11$ .

Then

$$\Delta_3(x) = S_{3,0}(12t + j) - S_{3,0}(12t + 8) =$$

$$\begin{cases} 0, & \text{if } j = 8, 9, \\ (-1)^{s_2(x-1)}, & \text{if } j = 10, \\ (-1)^{s_2(x-2)}, & \text{if } j = 11 \end{cases}$$

and (47) follows. ■

Now from (44)-(47) we easily deduce the following result.

**Theorem 3.**

$$(48) \quad S_{3,0}(x) = 3S_{3,0}(\lfloor \frac{x}{4} \rfloor) + \Delta_3(x) - (-1)^{s_2(\lfloor \frac{x}{4} \rfloor - 1)} A_3(x),$$

where  $A_3(x)$  and  $\Delta_3(x)$  are defined by (45) and (47) respectively.

Formula (48) gives a recursion for  $S_{3,0}(x)$ . Let us show that it coincides with the recursion (11)-(12), i.e.,

$$(49) \quad \Delta_3(x) - (-1)^{s_2(\lfloor \frac{x}{4} \rfloor - 1)} A_3(x) = \nu(x),$$

where  $\nu(x)$  is defined by (12). This follows from the following two lemmas.

**Lemma 4.** *The sequence*

$$(50) \quad \{(-1)^{s_2(x) + s_2(\lfloor \frac{x}{4} \rfloor - 1)} A_3(x)\}$$

*is periodic with period 8.*

*Proof.* In cases  $x \equiv i \pmod{8}$ ,  $i = 0, 1, 2, 3$  the terms of the sequence are zeros. If  $x \equiv i \pmod{8}$ ,  $i = 4, 5, 6, 7$ , put  $x = 8t + i$ . Then  $A_3(x) = 1$  and we have

$$(-1)^{s_2(x) + s_2(\lfloor \frac{x}{4} \rfloor - 1)} = (-1)^{s_2(8t+i) + s_2(2t)} =$$

$$(-1)^{s_2(8t+i) + s_2(8t)} = (-1)^{s_2(i)}$$

and the lemma follows. ■

Note that period of sequence (50) is



$$(51) \quad \{0, 0, 0, 0, -1, 1, 1, -1\}.$$

**Lemma 5.** *The sequence*

$$(52) \quad \{(-1)^{s_2(x)} \Delta_3(x)\}$$

*is periodic with period 12.*

*Proof.* According to (47), we have

$$(53) \quad \begin{cases} (-1)^{s_2(x)+s_2(x-1)}, & \text{if } x \equiv 1, 7 \text{ or } 10 \pmod{12} \\ (-1)^{s_2(x)+s_2(x-2)}, & \text{if } x \equiv 2 \text{ or } 11 \pmod{12} \\ (-1)^{s_2(x)+s_2(x-3)}, & \text{if } x \equiv 3 \pmod{12} \\ 0, & \text{otherwise.} \end{cases}$$

Let  $x = 12t + i$ ,  $0 \leq i \leq 11$ . Let, firstly,  $i = 1, 7, 10$ . In cases  $i = 1$  and  $i = 7$ , we, evidently, have  $(-1)^{s_2(x)+s_2(x-1)} = -1$ , while in case  $i = 10$ ,

$$(-1)^{s_2(12t+10)+s_2(12t+9)} = (-1)^{s_2(12t+1010_2)+s_2(12t+1001_2)} = 1.$$

Let now  $i = 2, 11$ . In case  $i = 2$ , we, evidently, have  $(-1)^{s_2(x)+s_2(x-2)} = -1$  and also in case  $i = 11$ , we find

$$(-1)^{s_2(12t+11)+s_2(12t+9)} = (-1)^{s_2(12t+1011_2)+s_2(12t+1001_2)} = -1;$$

finally, if  $i = 3$ , then, evidently, we have  $(-1)^{s_2(x)+s_2(x-3)} = 1$ . In other cases, the terms of the sequence are zeros. ■

Thus period of sequence (52) is

$$(54) \quad \{0, -1, -1, 1, 0, 0, 0, -1, 0, 0, 1, -1\}.$$

Subtracting the tripled period (51) from the doubled period (54), we obtain the period of length 24 of the left hand side of (49) multiplied by  $(-1)^{s_2(x)}$ . It is

$$(55) \quad \{0, -1, -1, 1, 1, -1, -1, 0, 0, 0, 1, -1, 1, -2, -2, 2, 0, 0, 0, -1, 1, -1, 0, 0\}.$$

It is left to note that, according to (12),  $(-1)^{s_2(x)} \nu(x)$  is periodic with the same period. ■

7. ON RECURSION FOR  $S_{n,0}(x)$ 

Let (42) be true. Let us write (42) for  $x := \lfloor \frac{x}{(n-1)^{n-1}} \rfloor$ . We have

$$(56) \quad \begin{aligned} & (-1)^{s_{n-1}(\lfloor \frac{x}{(n-1)^{n-1}} \rfloor - 1)} ((-1)^{\frac{n-1}{2}} S_{n,0}((n-1)^{n-1} \lfloor \frac{x}{(n-1)^{n-1}} \rfloor) + \\ & \sum_{j=0}^{\frac{n-3}{2}} (-1)^j \binom{n}{2j+1} S_{n,0}((n-1)^{2j} \lfloor \frac{x}{(n-1)^{n-1}} \rfloor)) = \\ & \begin{cases} 0, & \text{if } \lfloor \frac{x}{(n-1)^{n-1}} \rfloor \text{ is even,} \\ 1, & \text{if } \lfloor \frac{x}{(n-1)^{n-1}} \rfloor \text{ is odd.} \end{cases} \end{aligned}$$

Denote the left hand side of (56) by  $A_n(x)$ . Then, similar to (45), we have

$$(57) \quad \begin{aligned} & A_n(x) = \\ & \begin{cases} 0, & \text{if } x \equiv 0, \dots, (n-1)^{n-1} - 1, \pmod{2(n-1)^{n-1}}, \\ 1, & \text{if } x \equiv (n-1)^{n-1}, \dots, 2(n-1)^{n-1} - 1, \pmod{2(n-1)^{n-1}}. \end{cases} \end{aligned}$$

Furthermore, we consider the difference

$$(58) \quad \Delta_n(x) = S_{n,0}(x) - S_{n,0}((n-1)^{n-1} \lfloor \frac{x}{(n-1)^{n-1}} \rfloor).$$

**Lemma 6.**  $(-1)^{s_{n-1}(x)} \Delta_n(x)$  is periodic with period  $n(n-1)^{n-1}$ .

*Proof.* Indeed, let

$$x = n(n-1)^{n-1}t + j, \quad j = 0, 1, \dots, n(n-1)^{n-1} - 1.$$

Let  $j$  such that

$$\lfloor \frac{j}{(n-1)^{n-1}} \rfloor = m, \quad 0 \leq m \leq n-1.$$

Then

$$j = (n-1)^{n-1}m + k, \quad 0 \leq k \leq (n-1)^{n-1} - 1.$$

We have

$$\begin{aligned} \Delta_n(x) &= S_{n,0}(n(n-1)^{n-1}t + j) - S_{n,0}(n(n-1)^{n-1}t + (n-1)^{n-1}m) = \\ &= S_{n,0}(n(n-1)^{n-1}t + (n-1)^{n-1}m + k) - S_{n,0}(n(n-1)^{n-1}t + (n-1)^{n-1}m) = \end{aligned}$$

$$(59) \quad \sum_{i: (n-1)^{n-1}m+1 \leq 5i \leq (n-1)^{n-1}m+k-1} (-1)^{s_4(n(n-1)^{n-1}t+5i)}.$$

Note that

$$5i = (n-1)^{n-1}m + l, \quad 1 \leq l \leq k-1 \leq (n-1)^{n-1} - 2.$$

Therefore, the summands in (59) multiplied by  $(-1)^{s_{n-1}(x)}$  have the form

$$(-1)^{s_{n-1}(n(n-1)^{n-1}t+(n-1)^{n-1}m+k)+s_{n-1}(n(n-1)^{n-1}t+(n-1)^{n-1}m+l)}$$

and, since  $l < k \leq (n-1)^{n-1} - 1$ , this equal

$$(-1)^{s_{n-1}(n(n-1)^{n-1}t+(n-1)^{n-1}m)+s_{n-1}(k)+s_{n-1}(n(n-1)^{n-1}t+(n-1)^{n-1}m)+s_{n-1}(l)} = (-1)^{s_{n-1}(k)+s_{n-1}(l)}.$$

Therefore, the summands of (59) not depend on  $t$  and thus the sum (59), i.e.,  $\Delta_n(x)$  not depends on  $t$ . ■

**Lemma 7.** *The sequence*

$$(60) \quad \{(-1)^{s_{n-1}(x)+s_{n-1}(\lfloor \frac{x}{(n-1)^{n-1}} \rfloor - 1)} A_n(x)\}$$

*is periodic with period  $2(n-1)^{n-1}$ .*

*Proof.* In cases  $x \equiv i \pmod{2(n-1)^{n-1}}$ ,  $i = 0, 1, \dots, (n-1)^{n-1} - 1$  the terms of the sequence are zeros. If  $x \equiv i \pmod{2(n-1)^{n-1}}$ ,  $i = (n-1)^{n-1}, \dots, 2(n-1)^{n-1} - 1$ , put  $x = 2(n-1)^{n-1}t + i$ . Then  $A_n(x) = 1$  and we have

$$(-1)^{s_{n-1}(x)+s_{n-1}(\lfloor \frac{x}{(n-1)^{n-1}} \rfloor - 1)} = (-1)^{s_{n-1}(2(n-1)^{n-1}t+i)+s_{n-1}(2t)} = (-1)^{s_{n-1}(2(n-1)^{n-1}t+i)+s_{n-1}(2(n-1)^{n-1}t)} = (-1)^{s_{n-1}(i)}$$

and the lemma follows. ■

Now we obtain the following result.

**Theorem 8.** *If the conjectural relation (15) is true, then we have*

$$(61) \quad S_{n,0}(x) = \sum_{j=0}^{\frac{n-3}{2}} (-1)^{\frac{n-3}{2}-j} \binom{n}{2j+1} S_{n,0}((n-1)^{2j} \lfloor \frac{x}{(n-1)^{n-1}} \rfloor) + \nu_n(x),$$

where  $\nu_n(x)$  multiplied by  $(-1)^{s_{n-1}(x)}$  is periodic with period  $2n(n-1)^{n-1}$ .

*Proof.* Indeed, by (56)-(58), we obtain (61) with

$$\nu_n(x) = \Delta_n(x) + (-1)^{\frac{n-1}{2}+s_{n-1}(\lfloor \frac{x}{(n-1)^{n-1}} \rfloor - 1)} A_n(x).$$

Then, by Lemmas 6-7,  $(-1)^{s_{n-1}(x)} \nu_n(x)$  is periodic with period equal the least common multiple of numbers  $2(n-1)^{n-1}$  and  $n(n-1)^{n-1}$ . ■

As a corollary, in the case  $n = 3$  we again obtain Theorem 3 for  $\nu(x) = \nu_3(x)$  but without detailed representation of  $\Delta_3(x)$  and  $\nu(x)$ .

*Remark 9.* It follows from the proof that, if for some

$$j = j_i, \quad i = 1, \dots, k, \quad 1 \leq j_1 < j_2 < \dots < j_k \leq \frac{n-3}{2},$$

to replace in (61)  $S_{n,0}((n-1)^{2j} \lfloor \frac{x}{(n-1)^{n-1}} \rfloor)$  by  $S_{n,0}(\lfloor \frac{x}{(n-1)^{n-1-2j}} \rfloor)$  and to

denote the new sum by  $\Sigma(j_1, \dots, j_k)$ , then also the following form of Theorem 8 is valid

*Theorem 10. If the conjectural relation (15) is true, then we have*

$$(62) \quad S_{n,0}(x) = \Sigma(j_1, \dots, j_k) + \nu_n^{(j_1, \dots, j_k)}(x),$$

where  $\nu_n^{(j_1, \dots, j_k)}(x)$  multiplied by  $(-1)^{s_{n-1}(x)}$  is periodic with period  $2n(n-1)^{n-1}$ .

Thus we have  $2^{\frac{n-3}{2}}$  different formulas of type (62). In particular, in case  $n = 3$  we have only formula, in case  $n = 5$  we have two different formulas, etc.

#### 8. APPLICATION OF THEOREM 8 IN CASE $n = 5$

Since the conjectural identity (15) was proved in case  $n = 5$ , then, by Theorem 8, we conclude that

$$(63) \quad (-1)^{s_4(x)} \nu_5(x) = (-1)^{s_4(x)} (S_{5,0}(x) - 10S_{5,0}(\lfloor \frac{x}{256} \rfloor) + 5S_{5,0}(\lfloor \frac{x}{256} \rfloor))$$

is periodic with period 2560. If to write the period, then (63) gives a recursion for  $S_{5,0}(x)$ . The computer calculations show that the period with positions  $\{0, \dots, 2559\}$  contains all numbers from interval  $[-35, 35]$ . Here we give several sequences of positions in  $[0, 2559]$  with these numbers  $g \in [-35, 35]$ .

$$\begin{aligned} g = -35 &: \{251, 252, 254\}, \\ g = -34 &: \{246, 249, 1531, 1532, 1534\}, \\ g = -33 &: \{241, 243, 244, 1526, 1529\}, \\ g = -32 &: \{237, 239, 1521, 1523, 1524\}, \\ g = -31 &: \{231, 232, 234, 1517, 1519\}, \\ g = -30 &: \{197, 199, 200, 217, 219, 220, 226, 229, 511, 1511, 1512, 1514, \\ &\quad 2497, 2499, 2500, 2557, 2559\}, \\ &\quad \dots \\ g = 30 &: \{196, 198, 216, 218, 227, 228, 230, 1513, \\ &\quad 1515, 2496, 2498, 2556, 2558\}, \\ g = 31 &: \{233, 235, 1516, 1518, 1520\}, \\ g = 32 &: \{236, 238, 240, 1522, 1525\}, \\ g = 33 &: \{242, 245, 1527, 1528, 1530\}, \\ g = 34 &: \{247, 248, 250, 1533, 1535\}, \\ g = 35 &: \{253, 255\}. \end{aligned}$$

Besides, by Theorem 10, also

$$(64) \quad (-1)^{s_4(x)} \nu_5^{(1)}(x) = (-1)^{s_4(x)} (S_{5,0}(x) - 10S_{5,0}(\lfloor \frac{x}{16} \rfloor) + 5S_{5,0}(\lfloor \frac{x}{256} \rfloor))$$

is periodic with period 2560. Again, if to write the period, then (64) gives another recursion for  $S_{5,0}(x)$ . The computer calculations show that the period with positions  $\{0, \dots, 2559\}$  contains all numbers from interval  $[-9, 9]$ . Several sequences of positions in  $[0, 2559]$  with these numbers  $h \in [-9, 9]$  are the following:

$$h = -9 : \{2411, 2412, 2414, 2491, 2492, 2494\},$$

$$h = -8 : \{1131, 1132, 1134, 1211, 1212, 1214, 2406, 2409, 2486, 2489\},$$

...

$$h = 8 : \{1133, 1135, 1213, 1215, 2407, 2408, 2410, 2487, 2488, 2490\},$$

$$h = 9 : \{2413, 2415, 2493, 2495\}.$$

Finally, note that the sequence of the numbers of different values of  $\nu_3(x)$ ,  $\nu_5^{(1)}(x)$ ,  $\nu_5(x)$ , *etc.* begins with  $\{5, 19, 71, \dots\}$ .

### 9. RECURSIONS FOR $S_{3,1}(x)$ AND $S_{3,2}(x)$

Using (22)-(24), it is easy to show that the form  $3y(x) - y(4x)$  is invariant with respect to  $S_{3,i}(x)$ ,  $i = 0, 1, 2$ . This means that together with

$$(65) \quad 3S_{3,0}(x) - S_{3,0}(4x) = S_{3,0}(x) + S_{3,1}(x) + S_{3,2}(x),$$

we have also

$$(66) \quad 3S_{3,1}(x) - S_{3,1}(4x) = S_{3,0}(x) + S_{3,1}(x) + S_{3,2}(x),$$

$$(67) \quad 3S_{3,2}(x) - S_{3,2}(4x) = S_{3,0}(x) + S_{3,1}(x) + S_{3,2}(x).$$

Using (66)-(67), as in Section 6, we can prove that the expressions

$$(68) \quad (-1)^{s_2(x)} (S_{3,1}(x) - 3S_{3,1}(\lfloor \frac{x}{4} \rfloor)),$$

and

$$(69) \quad (-1)^{s_2(x)} (S_{3,2}(x) - 3S_{3,2}(\lfloor \frac{x}{4} \rfloor)),$$

are eventually priodic with the same period as  $(-1)^{s_2(x)} \nu(x)$  (12), i.e., the period (55), such that for  $S_{3,2}(x)$  the period starts at  $x = 8$ , while for  $S_{3,1}(x)$  the period starts at  $x = 16$ . This means that, for  $S_{3,i}(x)$ ,  $i = 1, 2$ , the same recursions hold as the recursion for  $S_{3,0}(x)$  (11) with the same function  $\nu(x)$  (12):

$$(70) \quad S_{3,1}(x) = 3S_{3,1}(\lfloor \frac{x}{4} \rfloor) + \nu(x), \quad x \geq 16,$$

with the initials

$$(71) \quad S_{3,1}(x) = \begin{cases} 0, & \text{if } x = 0, 1, \\ -1, & \text{if } x = 2, 3, 4, \\ -2, & \text{if } x = 5, 6, 7, 11, 12, 13, \\ -3, & \text{if } x = 8, 9, 10, 14, 15. \end{cases}$$

$$(72) \quad S_{3,2}(x) = 3S_{3,1}\left(\left\lfloor \frac{x}{4} \right\rfloor\right) + \nu(x), \quad x \geq 8,$$

with the initials

$$(73) \quad S_{3,2}(x) = \begin{cases} 0, & \text{if } x = 0, 1, 2, 6, 7, \\ -1, & \text{if } x = 3, 4, 5. \end{cases}$$

For example, by (70), (71) and (12), we have

$$S_{3,1}(20) = 3S_{3,1}(5) + \nu(20) = 3 \cdot (-2) + (-1)^{s_2(20)} = -5;$$

analogously, by (72), (73) and (12), we find

$$S_{3,2}(20) = 3S_{3,2}(5) + \nu(20) = 3 \cdot (-1) + (-1)^{s_2(20)} = -2.$$

## 10. A GENERALIZATION

A generalization of the conjectural equality (15) is the following

$$(74) \quad \sum_{j=0}^{\frac{n-1}{2}} (-1)^j \binom{n}{2j+1} S_{n,i}((n-1)^{2j}x) = \sum_{j=0}^{n-1} S_{n,j}(x), \quad i = 0, \dots, n-1, \quad x \geq 1, \quad n \equiv 1 \pmod{2}.$$

If this conjecture is valid, then, as in the previous sections, we can obtain the same recursions for every digit function  $S_{n,i}(x)$ ,  $i = 1, \dots, n-1$ , as for  $S_{n,0}(x)$  (cf. Theorems 8, 10). The question on initials in cases  $i \geq 1$  we here remain open.

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